

On the Question of the Point-Particle Nature of the Electron

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The electron and the positron treated as point particles in the Two Body Dirac equations of constraint dynamics for QED possess a new and as yet undiscovered peculiar 1S_0 bound-state which has a very large binding energy of about 300 keV, in addition to the usual 1S_0 positronium state with a binding energy of 6.8 eV. The production and detection of the peculiar 1S_0 state provide a test of the electron point-charge property. As the peculiar 1S_0 state lies lower than the usual 1S_0 state, the peculiar 1S_0 state can be produced by a two-photon decay of the usual 1S_0 state. We estimate the rate of the two-photon decay and show how it depends on the probability P_{up} of the admixture of the peculiar component in the predominantly usual 1S_0 positronium. The produced peculiar 1S_0 state in turn annihilates into two photons with a total c.m. energy of about 723 keV. Thus the signature for this new peculiar 1S_0 positronium bound state would be the decay of the usual 1S_0 state into four photons, with two energies bunching around 150 and two around 360 keV. Such a four-photon decay of the usual 1S_0 state will not be present if the electron and positron are not point particles, or if the mixing probability P_{up} is very small.

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I. INTRODUCTION

Quantum electrodynamics (QED) of light interacting with matter has had great successes and has been tested to a high degree of accuracy. As emphasized not the least by Dirac [1], Feynman [2], Jaffe [3], and many others, it is strange but apparently true that QED on its own is not mathematically consistent because it is not asymptotically free at short distances. Infinite charge and mass renormalizations are required at short distances, ensuring that the resulting perturbed masses and charges agree with observed values. From a fundamental point of view, QED raises the puzzling questions with regard to the meaning of equations involving infinite constants and mathematically undefined operations of infinite subtractions. Dirac [1] expresses the point of view that in analogy to the Bohr quantum theory, the present day QED formalism will probably be supplanted ultimately by a formalism that will not embody infinite charge and mass renormalizations. Thus the successes of the renormalization theory would then be seen to be on the same footing as the successes of the Bohr orbit theory applied to one-electron problems. Feynman himself, in a book published one year before he died [2], described the QED renormalization scheme in less kind words as a “hocus pocus” process that has prevented the proof that QED is mathematically consistent. In fairness to the memory of Feynman, however, it must be pointed out that when he learned that there was no Landau pole in QCD, he agreed that unlike QED, QCD is mathematically self-consistent. Perhaps in the hope that mathematicians may, as in times past with earlier problems, be able to rectify by more rigorous treatments on the problems of QED, Wightman [4] and later Jaffe [3, 5] developed and applied the methods of axiomatic field theory. The conclusion is that lacking asymptotic freedom, it is unlikely that QED could “be brought fully into the arena of mathematics” [3]. This still leaves however Dirac’s basic objection related to the appearances of the infinite subtractions.

In the classical theory of electrodynamics, the self-energy is infinite for a point particle. Weinberg has pointed out that the infinite classical self-energy for a point particle should be taken as a warning of similar problems to come in the nature of infinity subtractions of the masses and charges for particles in quantum field theory [6]. The point nature of particles is therefore intimately related to the infinite self-energy and the question of renormalization. Is it necessary to introduce unknown interactions or unknown electron structure to remove the ambiguities arising from the infinities, as was suggested in early classical models by Abraham [7] and later Dirac [8], and the axiomatic field theory of Wightman [4] and Jaffe [3, 5]? If so, what types of interactions and electron structure will be needed to examine such a problem¹? Is it alternatively necessary to bypass the problem of infinite self-energies by postulating fields which only act on other particles and only by action-at-a-distance [11]? One can also avoid classical point-particle mass and charge singularities by using Wheeler’s geometrodynamical wormhole descriptions of a “charge without charge” and a “mass without mass” [12–14].

Rather than focus on theoretical models related to the point-particle nature, we look for observable properties of e^+e^- bound states that may depend on this point-particle property of the constituents. It is worth pointing out that the concept of an electron point charge has been commonly assumed. From the close agreement of experimental and theoretical electron g -values, an upper limit for the electron radius of 10^{-17} m, may be extracted [15]. It is reinforced by the absence of a form factor in high-energy electron scattering measurements with an upper limit of the interaction distance scale of order 3×10^{-19} m [16], and the small magnitude of the upper limit of the electron dipole moment, $d_e < 8.7 \times 10^{-29}$ e·cm [17, 18]. It is therefore reasonable to examine the consequences of the point electron concept and look for related physical observables that may be probed by experimental measurements.

It is clear that the point charge property has the greatest effects on the interaction between the electron and the positron at short distances. In this regard, we note that the magnetic hyperfine interaction between the electron and the positron in the S state is given by Eq. (5.73) of [19],

$$H_{\text{HFS}} = -\frac{8\pi}{3} \boldsymbol{\mu}_{e^-} \cdot \boldsymbol{\mu}_{e^+} \delta(\mathbf{r}), \quad (1)$$

where the magnetic moments $\boldsymbol{\mu}_{e^\pm}$ of e^+ and e^- are related to their spins \mathbf{s}_{e^\pm} by $\boldsymbol{\mu}_{e^\pm} = e^\pm \mathbf{s}_{e^\pm} / mc$. In the spin-singlet 1S_0 state for which both e^+e^- and $\mathbf{s}_{e^+} \cdot \mathbf{s}_{e^-}$ are negative, the above spin-spin interaction is attractive and singular at short distances and may lead to observable point-charge effects in e^+e^- bound states. The traditional treatment of the above interaction presumes the usual positronium of radius $1/\alpha m$ and treats the interaction as a perturbation. It does not touch upon nonperturbative bound-state effects that can be investigated only by solving the bound-state equation with the inclusion of the spin-spin interaction. However, the attractive delta-function interaction of Eq. (1) is too singular to be solved if it is included in the non-relativistic Schrödinger equation. Therefore, the presence of the strongly attractive spin-spin interaction in the 1S_0 state necessitates a proper nonperturbative relativistic treatment of the two-body bound-state problem.

¹ In contrast, Arnowitz, Deser, and Misner have found that only in the limit of a point charge and mass do gravitational forces exactly counteract the repulsive electrostatic self-forces giving stable and static charged point particle configurations [9]. See also [10].

The Two Body Dirac Equations (TBDE) of Dirac's constraint dynamics have been previously tested and found to be a proper formalism to study relativistic two-body bound states. In QCD, they lead to a good relativistic description of meson spectroscopy in terms of quark-antiquark bound states for both light and heavy mesons [20–23]. In QED, they yield not only a perturbative spectra that agree with QED standard results but also distinguish themselves from other bound-state approaches in their ability to reproduce these same spectral results by nonperturbative bound-state methods, both numerically and analytically [20, 24, 25]. They give the singular spin-spin interaction of Eq. (1) in the non-relativistic limit. It is therefore appropriate to use the TBDE to study point-charge effects in e^+e^- bound states.

Using the TBDE equations, we found point-charge effects which appears as the presence of new positronium bound states, in addition to the usual positronium states [26]. Their origin was made clear by the Schrödinger-like equation that comes from the Pauli reduction of the TBDE (see Eq. (8) below). In particular the magnetic spin-spin interaction in the 1S_0 states² is indeed very attractive at short distances, modified by the relativistic structure of the equations to become less singular in relativistic constraint dynamics than the delta function in the non-relativistic approximation³. However, in the 1S_0 state, the magnetic interaction exactly cancels the very repulsive Darwin interaction, resulting in a quasipotential that behaves as $-\alpha^2/r^2$ near the origin, for point electron and positron. The bound-state equation then admits two different types of states which we designate as usual and peculiar. They possess distinctly different properties at short distances. In particular, the peculiar 1S_0 state, yet to be observed, has a root-mean-square radius of approximately $1/m$ and a rest mass approximately $\sqrt{2}m$, in contra-distinction from the usual 1S_0 state with a root-mean-square radius of $1/(\alpha m)$ and a mass of approximately $2m - m\alpha^2/4$.

The existence of the usual and peculiar states for the positronium system poses conceptual and mathematical problems [26]. If we keep both sets of states in the same Hilbert space, then each set is complete by itself, but the two sets of states are not orthogonal to each other. Our system is thus over-complete. Furthermore, the matrix element of the scaled invariant mass operator for these states between states of one type and the states of the other type are not symmetric and thus the invariant mass operator is not self-adjoint.

With the quasipotential $-\alpha^2/r^2$ at short distances for the 1S_0 state as it has been determined by the TBDE constraint dynamics, both the usual and peculiar states are physically admissible and there do not appear to be compelling reasons to exclude one of the two sets as being unphysical. We were therefore motivated to introduce a “peculiarity” quantum number ζ , such that $\zeta = +1$ for usual states that have properties the same as those one usually encounters in QED, and $\zeta = -1$ for peculiar states. The introduction of the peculiarity quantum number enlarged the Hilbert space to contain both usual and peculiar states in a complete set and made the mass operator self-adjoint.

It should be emphasized that if the electron is not a point particle, then the peculiar state will not exist. Therefore, an experimental search of the peculiar states can be used to find out whether the electron is a point particle or not. As the usual 1S_0 state of mass $\sim 2m$ lies above the peculiar state of mass $\sim \sqrt{2}m$, the usual 1S_0 state can decay into the peculiar 1S_0 state by a $0^+ \rightarrow 0^+$ transition, with the emission of two photons. Because such a decay has not yet been observed, it is reasonable to consider the usual 1S_0 state to be predominantly a peculiarity $\zeta=1$ state, with a small admixture amplitude $M_{\zeta\zeta'}$ of the peculiarity $\zeta'=-1$ component. Through its admixture to the peculiar sector, a state in the usual sector can decay to a state of lower energy in the peculiar sector. The mixing probability $P_{up} = |M_{\zeta\zeta'}|^2$, for the usual 1S_0 state to admix with the peculiar 1S_0 state, can be determined by measuring the decay rate of (usual 1S_0) \rightarrow (peculiar 1S_0) $+ 2\gamma$.

To assist the determination of the mixing probability, we would like to evaluate how the two-photon decay rate from the usual 1S_0 state to the peculiar 1S_0 state depend on P_{up} . After its production, the 1S_0 peculiar state will promptly annihilate into two photons. We would like to calculate the rate of annihilation of the 1S_0 peculiar state and identify the signature for the production of the peculiar state.

In the next section we review the formalism leading to the usual and peculiar solutions of the TBDE for the 1S_0 state of positronium. In section 3 we obtain an estimate for the decay rate of the usual 1S_0 state to undergo a meta-stable two-photon decay into the peculiar 1S_0 state. In section 4 we evaluate the annihilation lifetime of the peculiar 1S_0 state. In section 5, we present the conclusions and discussions. Relevant details are given in the Appendix.

II. USUAL AND PECULIAR BOUND STATE SOLUTIONS

The Two-Body Dirac equations of constraint dynamics give a manifestly covariant 3D reduction of the Bethe-Salpeter equation for two spin-1/2 particles [27]. It provides a route [28] around the Currie-Jordan-Sudarshan

² The spin and orbital quantum numbers of the 1S_0 state refer to those of the ψ_+ wave function in Eq. (8), a four component subset of the full sixteen component spinor.

³ For an explicit expression of the relativistic spin-spin interaction in the 1S_0 state, see Eq. (21) of [26].

“non-interaction theorem” [29] that apparently forbade canonical 4-dimensional treatment of the relativistic N -body problem. For two particles interacting through a vector interactions the TBDE are given by

$$\mathcal{S}_1\psi \equiv \gamma_{51}(\gamma_1 \cdot (p_1 - \tilde{A}_1) + m_1)\psi = 0, \quad (2a)$$

$$\mathcal{S}_2\psi \equiv \gamma_{52}(\gamma_2 \cdot (p_2 - \tilde{A}_2) + m_2)\psi = 0, \quad (2b)$$

in which ψ is a 16 component spinor. The operators are compatible with

$$[\mathcal{S}_1, \mathcal{S}_2]\psi = 0, \quad \text{implying } \tilde{A}_i = \tilde{A}_i(x_\perp). \quad (3)$$

Thus the potential is forced to depend on $x_1 - x_2$ only through the transverse component

$$\begin{aligned} x_\perp^\mu &= \left(\eta^{\mu\nu} + \hat{P}^\mu \hat{P}^\nu \right) (x_1 - x_2)_\nu, \quad x_\perp \cdot \hat{P} = 0, \\ P &= p_1 + p_2, \\ \hat{P} &= P/\sqrt{-P^2}. \end{aligned} \quad (4)$$

One can further show from these two constraints that

$$p \cdot P\psi = 0. \quad (5)$$

Thus, in the center-of-momentum (c.m.) frame where

$$P = (w, \mathbf{0}), \quad (6)$$

the relative energy is eliminated ($p\psi = (0, \mathbf{p})\psi$) and the relative time does not appear ($x_\perp = (0, \mathbf{r})$). The compatibility condition, $[\mathcal{S}_1, \mathcal{S}_2]\psi = 0$, also restricts the spin dependence of \tilde{A}_i^μ by determining their dependence on γ_1, γ_2 , [26]

$$\tilde{A}_i^\mu = \tilde{A}_i^\mu(A(r), p_\perp, \hat{P}, w, \gamma_1, \gamma_2), \quad (7)$$

with vector interactions \tilde{A}_i^μ that depend on an invariant $A(r)$ through the vertex form of $\gamma_1 \cdot \gamma_2$. The Pauli reduction of the TBDE leads to a covariant Schrödinger-like equation for relative motion with an explicit spin-dependent potential Φ . In the c.m. system it takes the form

$$\begin{aligned} \mathcal{B}^2\psi_+ &\equiv \{\mathbf{p}^2 + \Phi(\mathbf{r}, m_1, m_2, w, \sigma_1, \sigma_2, \mathbf{L})\}\psi_+ \\ &= \left\{ \mathbf{p}^2 + 2\varepsilon_w A - A^2 + \Phi_D + \sigma_1 \cdot \sigma_2 \Phi_{SS} \right. \\ &\quad + \mathbf{L} \cdot (\sigma_1 + \sigma_2) \Phi_{SO} + (3\sigma_1 \cdot \hat{\mathbf{r}} \sigma_2 \cdot \hat{\mathbf{r}} - \sigma_1 \cdot \sigma_2) \Phi_T \\ &\quad + \mathbf{L} \cdot (\sigma_1 - \sigma_2) \Phi_{SOD} + i\mathbf{L} \cdot \sigma_1 \times \sigma_2 \Phi_{SOX} \\ &\quad \left. + \sigma_1 \cdot \hat{\mathbf{r}} \sigma_2 \cdot \hat{\mathbf{r}} \mathbf{L} \cdot (\sigma_1 + \sigma_2) \Phi_{SOT} \right\} \psi_+ \\ &= b^2\psi_+, \end{aligned} \quad (8)$$

where ψ_+ is a 4-component spinor subcomponent of 16 component spinor ψ . The quasipotentials $\Phi_D, \Phi_{SS}, \Phi_{SO}, \Phi_T, \Phi_{SOD}, \Phi_{SOX}$, and Φ_{SOT} correspond to the Darwin, spin-spin, spin-orbit, tensor, spin-orbit difference, spin-orbit product, and spin-orbit tensor interactions, respectively. Explicit expressions of these interactions are given in [26]. The kinematical variables

$$m_w = \frac{m_1 m_2}{w}, \quad (9)$$

$$\varepsilon_w = \frac{w^2 - m_1^2 - m_2^2}{2w}, \quad (10)$$

satisfy

$$b^2 = \varepsilon_w^2 - m_w^2 = \frac{1}{4w^2} [w^4 - 2w^2(m_1^2 + m_2^2) + (m_1^2 - m_2^2)^2], \quad (11)$$

which corresponds to the Einstein relation between the energy and reduced mass for the fictitious particle of relative motion. The effects of an eikonal approximation of all the ladder and cross ladder diagrams and iterated constraint diagrams are embedded in the c.m. energy dependencies seen in Eq. (8) [30]. In [26] a number of properties of the

TBDE are listed (see also [23]). Among those that will be of importance here is that the TBDE provide a covariant 3D framework in which the local potential approximation consistently fulfills the requirements of gauge invariance in QED [31] and that the Schrödinger-like equation with $\Phi(A = -\alpha/r)$ is responsible for accurate QED spectral results through order α^4 [25].

The QED spectral results can be obtained by solving the radial forms of Eq. (8), either numerically or analytically. For equal-mass systems of e^+e^- in the 1S_0 state, the spin-spin interaction $-3\Phi_{SS}$ is indeed very attractive and strong at short distances, behaving as $-9/8r^2$ (this follows from Eq. (21) in [26]). Although this is not as singular as its more well known non-relativistic delta-function form given in Eq. (1), by itself it would be regarded as singular since it is more attractive than $-1/4r^2$ and would prevent a nonperturbative treatment of this term. However, for equal-mass systems such as e^+e^- in the 1S_0 state, there is an exact cancellation of the complete spin-spin term $-3\Phi_{SS}$ with the highly repulsive Darwin interaction Φ_D [26], resulting in a quasipotential that behaves as $-\alpha^2/r^2$ near the origin. The bound-state equation can thus be treated nonperturbatively. The eigenvalue equation for the 1S_0 state becomes

$$\left\{-\frac{d^2}{dr^2} + 2\varepsilon_w A - A^2\right\}u_0 = b^2 u_0. \quad (12)$$

For a point electron and positron with $A = -\alpha/r$, the above becomes

$$\left\{-\frac{d^2}{dr^2} - \frac{2\varepsilon_w \alpha}{r} - \frac{\alpha^2}{r^2}\right\}u_0 = b^2 u_0. \quad (13)$$

We can examine the behavior of the wave function at short distances ($r \ll \alpha/2\varepsilon_w$), where the above equation behaves as

$$\left\{-\frac{d^2}{dr^2} - \frac{\alpha^2}{r^2}\right\}u_0 = 0. \quad (14)$$

Such a short-distance limit is independent of the chosen gauge [31]. The indicial equation has two types of solutions which will be called usual and peculiar,

$$\begin{aligned} u_+ &\sim r^{\lambda_+ + 1}; \quad \lambda_+ = (-1 + \sqrt{1 - 4\alpha^2})/2; \quad + \text{ usual} \\ u_- &\sim r^{\lambda_- + 1}; \quad \lambda_- = (-1 - \sqrt{1 - 4\alpha^2})/2; \quad - \text{ peculiar}. \end{aligned} \quad (15)$$

At short distances, the probability is *finite* for solutions of both types,

$$\psi_{\pm}^2 d^3r = \frac{u_{\pm}^2}{r^2} r^2 dr d\Omega = u_{\pm}^2 dr d\Omega = r^{(1 \pm \sqrt{1 - 4\alpha^2})} dr d\Omega, \quad (16)$$

which indicates that the behaviors of the wave functions of both types are quantum mechanically acceptable near the origin. If $L \neq 0$ so that $L(L+1) - \alpha^2 > 0$ or if the electron is not a point particle, then the peculiar solution is not physically admissible⁴.

Both 1S_0 bound state solutions can be obtained analytically. The respective sets of eigenvalues for total invariant center-of-mass energy (mass) $w_{\pm n}$ (n is the principle quantum number) are [26]

$$w_{\pm n} = m\sqrt{2 + 2/\sqrt{1 + \alpha^2/(n \pm \sqrt{1/4 - \alpha^2} - 1/2)^2}}. \quad (17)$$

The eigenvalue of a usual state is obtained by taking the positive sign of the above equation. Its expansion in powers of α gives the standard QED perturbative results through order α^4

$$w_{+n} = 2m - m\alpha^2/4n^2 - m\alpha^4/2n^3(1 - 11/32/n) + O(\alpha^6), \quad n = 1, 2, 3, \dots \quad (18)$$

For the usual ground ($n = 1$) state, it gives $w_{+n} \sim 2m - m\alpha^2/4$.

The eigenvalue of the peculiar ($n = 1$) ground state is obtained by taking the negative sign of Eq. (17). It has a mass

$$w_{-1} = m\sqrt{2 + 2/\sqrt{1 + \alpha^2/(1/2 - \sqrt{1/4 - \alpha^2})^2}} \sim \sqrt{2}m\sqrt{1 + \alpha}, \quad (19)$$

⁴ In Schiff's Quantum Mechanics [32], a solution similar to the peculiar one discussed here is examined for the case of the Klein Gordon equation for the Coulomb system. He argues that what we would call the peculiar solution can be discarded since the source of the Coulomb attraction is a finite sized nucleus of radius r_0 . Here the issue is not resolved in that way since we are allowing for the possibility that the electron and positron are point particles in order to test the consequences of that assumption.

which represents very tight binding energy on order 300 keV for an e^+e^- state. The size of the peculiar ground state is on the order of a Compton wave length $1/m$ [26], much smaller than the Bohr radius size of the usual positronium ground state. Its weak coupling limit has a total c.m. energy approximately $\sqrt{2}m$ instead of $2m$. We point out here that this solution does not have the usual non-relativistic limit.

The two $n = 1$ wave functions have the respective forms,

$$\begin{aligned} u_+(r) &= c_+ r^{\lambda_++1} \exp(-\kappa_+ \varepsilon_{w_+} \alpha r), \quad \kappa_+ = \frac{2}{1 + \sqrt{1 - 4\alpha^2}} = \frac{1}{\lambda_+ + 1}, \\ u_-(r) &= c_- r^{\lambda_-+1} \exp(-\kappa_- \varepsilon_{w_-} \alpha r), \quad \kappa_- = \frac{2}{1 - \sqrt{1 - 4\alpha^2}} = \frac{1}{\lambda_- + 1}. \end{aligned} \quad (20)$$

Since they are both zero node solutions, they are not orthogonal (although the inner product is small, $\sim 1/1000$)

$$\langle u_- | u_+ \rangle = \int_0^\infty dr u_+(r) u_-(r) \sim \alpha^{3/2} \neq 0. \quad (21)$$

How do we reconcile this with expected orthogonality of the eigenfunctions of a self-adjoint operator corresponding to different eigenvalues? One can show that the second derivative is not self-adjoint [26] in this context! However, we emphasize the fact that both sets of usual and peculiar states are quantum mechanically admissible states. We admit both types of physical states into a larger Hilbert space by introducing a new operator $\hat{\zeta}$ with observable quantum number ζ , which we call ‘‘peculiarity’’. This will allow the mass operator to be self-adjoint, and the set of physically allowed states to become a complete set. In particular we let

$$\begin{aligned} \hat{\zeta} \chi_+ &= \zeta \chi_+ = +\chi_+ \quad \text{with eigenvalue } \zeta = +1, \text{ usual positronium,} \\ \hat{\zeta} \chi_- &= \zeta \chi_- = -\chi_+ \quad \text{with eigenvalue } \zeta = -1, \text{ peculiar positronium,} \end{aligned} \quad (22)$$

with the corresponding spinor wave function χ_ζ assigned to the states so that a usual state is represented by the peculiarity spinor χ_+ ,

$$\chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad (23)$$

and a peculiar state is represented by the peculiarity spinor χ_-

$$\chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad (24)$$

With this introduction, a general wave function can be expanded in terms of the complete set of basis functions $\{u_{+n}, u_{-n}\}$ as

$$\Psi = \sum_{\zeta n} a_{\zeta n} u_{\zeta n} \chi_\zeta, \quad (25)$$

where n represents spin and spatial quantum numbers and ζ the peculiarity. The variational principle applied to \mathcal{B}^2

$$\langle \mathcal{B}^2 \rangle = \frac{\langle \Psi | \mathcal{B}^2 | \Psi \rangle}{\langle \Psi | \Psi \rangle}, \quad (26)$$

would lead to

$$\begin{aligned} \mathcal{B}^2 u_{+n} \chi_+ &= -\kappa_{+n}^2 u_{+n} \chi_+, \\ \mathcal{B}^2 u_{-n} \chi_- &= -\kappa_{-n}^2 u_{-n} \chi_-. \end{aligned} \quad (27)$$

Thus the introduction of the peculiarity quantum number resolves the problem of the over-completeness property of the basis states and the non-self-adjoint property of the mass operator \mathcal{B}^2 .

For completeness, it is worth pointing out that peculiar states similar to those described above for e^+e^- would appear also in other point-charge equal-mass fermion systems such as $\mu^+\mu^-$. However for unequal-mass fermion systems, the repulsive Darwin interaction overwhelms the attractive spin-spin interaction as short distance. There would thus be no attractive $-\alpha^2/r^2$ term and no peculiar sector. The Darwin interaction Φ_D has dual origins: the retardative effects and the usual zitterbewegung blurring of the relative coordinate such as appears for the electron in the hydrogen atom. The former part has been evaluated for (spin-zero)-(spin-zero) bound states (e.g. $\pi^+\pi^-$) and shown to be repulsive [33]. The latter part is also repulsive but is absent for (spin-zero)-(spin-zero) bound states.

III. PRODUCTION OF THE PECULIAR 1S_0 STATE

If the peculiar sector and the usual section are disconnected, the peculiarity quantum number is strictly conserved and states of one sector will not make transitions to the other sector. There would be no way to produce the peculiar states from the usual states. The usual ground 1S_0 state would only undergo the usual two photon annihilation in about 10^{-10} sec as shown in Fig. 1.

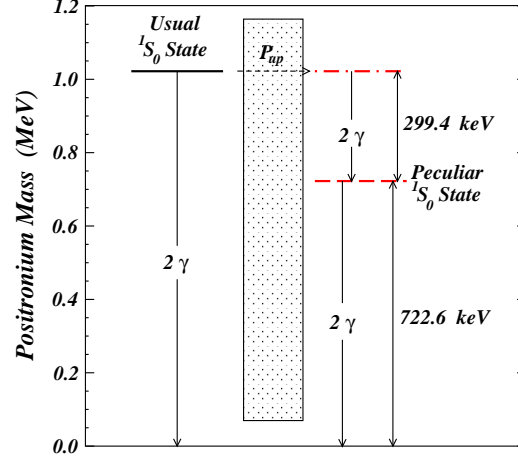


FIG. 1. (Color online) Schematic energy level diagram for the production and detection of the peculiar 1S_0 state. The usual positronium normally decays by annihilation with the emission of 2 photons. Through its admixture to the peculiar sector with an admixture probability P_{up} , the usual 1S_0 state can decay to the peculiar 1S_0 state at $w \sim \sqrt{2}m$ with the emission of two photons. The peculiar 1S_0 state will subsequently annihilate into two additional photons.

We envisage however that while these two sectors are distinct, the peculiar quantum number may not be conserved for the full Hamiltonian, and a physical state is an admixture of a usual state and a peculiar state. The physical 1S_0 state, that is predominantly a usual $\zeta=1$ state, may be presumed to be $\sqrt{1 - |M_{\zeta\zeta'}|^2}\chi_+ + M_{\zeta\zeta'}\chi_-$ with an mixing probability $P_{up} = |M_{\zeta\zeta'}|^2$ in the peculiar sector. Through its admixture to the peculiar sector, a state in the usual sector can decay to a state in the peculiar sector with a lower energy. In that case the higher (predominantly) usual (1^1S_0) state located at $w_{+1} \sim 2m - m\alpha^2/4$ can undergo a meta-stable $0^+ \rightarrow 0^+$ decay into the lower (predominantly) peculiar (1^1S_0) state located at $w_{-1} \sim \sqrt{2}m$ by emitting two photons, as shown in Fig. 1. The subsequent annihilation of the peculiar 1^1S_0 state will result in two additional photons for a total of four photons, with two energies bunching around 150 and two around 360 keV. The signals of 4 photon decays of definite energies thus constitute the signature for the peculiar state. The rate of the peculiar state production allows the determination of the mixing probability P_{up} .

For brevity of notations, we shall abbreviate the usual ground (1^1S_0) state by $1S_u$ and the peculiar ground (1^1S_0) state by $1S_p$, where the $2s+1$ superscript index and the J subscript index are made implicit except when they are needed to resolve ambiguities. Having assumed the mixing probability P_{up} , we proceed to determine how the rate of production of the peculiar $1S_p$ state through $1S_u \rightarrow 1S_p + 2\gamma$ depends on P_{up} .

With the decaying usual ground spin-singlet state $1S_u$ initially at rest, the decay of the usual $1S_u$ state is a three-body decay. The produced peculiar $1S_p$ state would experience some recoil from the metastable decay and would have a differential transition rate of

$$d\Gamma(1S_u \rightarrow 1S_p + 2\gamma) = 2\pi |T_{1S_u, 1S_p + 2\gamma}|^2 d^3k_1 d^3k_2 d^3p_{1S_p} \delta(E_{1S_u} - E_{1S_p} - \hbar\omega_1 - \hbar\omega_2) \delta(\mathbf{0} - \mathbf{p}_{1S_p} - \mathbf{k}_1 - \mathbf{k}_2), \quad (28)$$

where $T_{1S_u, 1S_p + 2\gamma}$ is the transition amplitude for the process $1S_u \rightarrow 1S_p + 2\gamma$ with the emission of photons characterized by (\mathbf{k}_1, α_1) and (\mathbf{k}_2, α_2) . We work in the c.m. system of the initial positronium atom and use

$$E_{1S_p} = \sqrt{(\sqrt{2}m)^2 + \mathbf{p}_{1S_p}^2}. \quad (29)$$

Performing the $d^3p_{1S_p}$ integral gives

$$d^2\Gamma(1S_u \rightarrow 1S_p + 2\gamma) = 2\pi |T_{1S_u, 1S_p + 2\gamma}(\omega_1, \omega_2)|^2 d^3k_1 d^3k_2 \delta(2m - E_{1S_p} - \omega_1 - \omega_2), \quad (30)$$

in which

$$\begin{aligned} E_{1S_p} &= \sqrt{2m^2 + \omega_1^2 + \omega_2^2 + 2\omega_1\omega_2 \cos \theta_{12}}, \\ \cos \theta_{12} &= \cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2 \cos(\phi_1 - \phi_2). \end{aligned} \quad (31)$$

We use perturbation theory to evaluate the transition matrix element $T_{1S_u \rightarrow 1S_p+2\gamma}$. The perturbative interaction leading to the transition is determined by considering Eqs. (2) and (8) and adding an external quantized space-like photon vector potentials $A_\perp(x_i)$ orthogonal to P , in order to define an effective decay Hamiltonian for our two-body system. The minimal substitution in Eq. (2) leads to

$$\begin{aligned} \gamma_{51}(\gamma_1 \cdot (p_1 - \tilde{A}_1 - eA_\perp(x_1)) + m_1)\psi &= 0, \\ \gamma_{52}(\gamma_2 \cdot (p_2 - \tilde{A}_2 + eA_\perp(x_2)) + m_2)\psi &= 0. \end{aligned} \quad (32)$$

The problem of compatibility of the two separate Dirac equations, without the presence of an external potential, has been solved and its Pauli reduction leads to Eq. (8). We can determine the form that Eq. (8) would subsequently take on if the space-like parts of the constituent momenta are modified by minimal substitutions.

In order to set up the decay Hamiltonian in the relativistic case of a two body system with opposite charges we note that the manifestly invariant form for Eq. (8) is⁵

$$[p_\perp^2 - b^2(w) + \Phi(x_\perp, w, \sigma_1, \sigma_2, L)] \psi_+. \quad (33)$$

We use the constituent c.m. energies $\varepsilon_1/w = \varepsilon_2/w = 1/2$. With $m_1 = m_2 \equiv m$, $b^2(w) = w^2/4 - m^2$, $P \cdot p\psi_+ = 0$, we have

$$\begin{aligned} (p_\perp^2 - b^2(w)) \psi_+ &= (p^2 - b^2(w)) \psi_+ = \frac{1}{2} \left(\left(\frac{P}{2} + p \right)^2 + m_1^2 \right) + \frac{1}{2} \left(\left(\frac{P}{2} - p \right)^2 + m_2^2 \right) \psi_+ \\ &= \left(\frac{1}{2} (p_1^2 + m_1^2) + \frac{1}{2} (p_2^2 + m_2^2) \right) \psi_+. \end{aligned} \quad (34)$$

We extend this minimally by

$$\begin{aligned} p_1 &\rightarrow p_1 - eA_\perp(x_1), \\ p_2 &\rightarrow p_2 + eA_\perp(x_2). \end{aligned} \quad (35)$$

This leads to

$$\begin{aligned} p_\perp^2 - b^2(w) &\rightarrow p_\perp^2 - b^2(w) \\ &\quad - \frac{1}{2} [e(p_\perp \cdot A_\perp(x_1) + A_\perp(x_1) \cdot p_\perp) + e^2 A_\perp^2(x_1)] \\ &\quad - \frac{1}{2} [e(p_\perp \cdot A_\perp(x_2) + A_\perp(x_2) \cdot p_\perp) + e^2 A_\perp^2(x_2)]. \end{aligned} \quad (36)$$

In the c.m. system ($A_\perp = (0, \mathbf{A})$) then

$$\begin{aligned} \mathbf{p}^2 - b^2(w) + \Phi &\rightarrow \mathbf{p}^2 - b^2(w) + \Phi \\ &\quad - \frac{1}{2} [e(\mathbf{p} \cdot \mathbf{A}(\mathbf{x}_1, t) + \mathbf{A}(\mathbf{x}_1, t) \cdot \mathbf{p}) + e^2 \mathbf{A}^2(\mathbf{x}_1)] \\ &\quad - \frac{1}{2} [e(\mathbf{p} \cdot \mathbf{A}(\mathbf{x}_2, t) + \mathbf{A}(\mathbf{x}_2, t) \cdot \mathbf{p}) + e^2 \mathbf{A}^2(\mathbf{x}_2)]. \end{aligned} \quad (37)$$

⁵ The four vector forms of the spin and orbital angular momenta are $\sigma_i^\mu = \varepsilon^{\mu\nu\kappa\lambda} \sigma_{i\nu\kappa} \hat{P}_\lambda$, $L^\mu = \varepsilon^{\mu\nu\kappa\lambda} x_{\perp\nu} p_{\kappa} \hat{P}_\lambda$.

We define

$$\begin{aligned}
H &= \frac{1}{2\mu} \mathbf{p}^2 + \frac{1}{2\mu} \Phi \\
&\quad - \frac{1}{4\mu} [e (\mathbf{p} \cdot \mathbf{A}(\mathbf{x}_1, t) + \mathbf{A}(\mathbf{x}_1, t) \cdot \mathbf{p}) + e^2 \mathbf{A}^2(\mathbf{x}_1)] \\
&\quad - \frac{1}{4\mu} [e (\mathbf{p} \cdot \mathbf{A}(\mathbf{x}_2, t) + \mathbf{A}(\mathbf{x}_2, t) \cdot \mathbf{p}) + e^2 \mathbf{A}^2(\mathbf{x}_2)] \\
H &= H_0 + H_{int}, \\
H_{int} &= -\frac{1}{4\mu} [e (\mathbf{p} \cdot \mathbf{A}(\mathbf{x}_1, t) + \mathbf{A}(\mathbf{x}_1, t) \cdot \mathbf{p}) + e^2 \mathbf{A}^2(\mathbf{x}_1)] \\
&\quad - \frac{1}{4\mu} [e (\mathbf{p} \cdot \mathbf{A}(\mathbf{x}_2, t) + \mathbf{A}(\mathbf{x}_2, t) \cdot \mathbf{p}) + e^2 \mathbf{A}^2(\mathbf{x}_2)].
\end{aligned} \tag{38}$$

Our desired matrix element is

$$T_{1S_u, 1S_p+2\gamma} = \langle 1S_p \gamma \gamma | H_{int} | 1S_u \rangle M_{\zeta\zeta'}, \tag{39}$$

where $M_{\zeta\zeta'}$ is related to the mixing probability P_{up} by $P_{up} = |M_{\zeta\zeta'}|^2$. In Appendix A we find in the dipole approximation that

$$T_{1S_u, 1S_p+2\gamma} \simeq \left(\frac{e^2}{m} \frac{1}{2(2\pi)^3 \sqrt{\omega_1 \omega_2}} \right) \left[2^{5/4} \alpha^{3/2} \right] M_{\zeta\zeta'}, \tag{40}$$

and so

$$|T_{1S_u, 1S_p+2\gamma}(\omega_1, \omega_2)|^2 \sim P_{up} \left| \frac{e^2}{m} \frac{1}{2(2\pi)^3 \sqrt{\omega_1 \omega_2}} 2^{5/4} \alpha^{3/2} \right|^2. \tag{41}$$

Appendix A shows that the above transition amplitude leads to the lifetime for the transition of $1S_u \rightarrow 1S_p + 2\gamma$ as

$$\tau(1S_u \rightarrow 1S_p + 2\gamma) \sim \frac{\tau_{1S_u \rightarrow 2\gamma}}{0.152 P_{up}} = \frac{8.2 \times 10^{-10}}{P_{up}} \text{ sec}, \tag{42}$$

corresponding to a branching ratio of

$$P(1S_u \rightarrow 2\gamma) : P(1S_u \rightarrow 1S_p + 2\gamma) = 1 : 0.152 P_{up}. \tag{43}$$

IV. ANNIHILATION OF THE PECULIAR 1S_0 STATE INTO TWO PHOTONS

After the peculiar $1S_p$ state is produced, it will subsequently annihilate into two photons with an energy of approximately 360 keV each. In [35] we presented previously a formalism for positronium annihilation, especially suited for relativistic wave functions that have mild singularities at the origin as occurs with our usual and peculiar wave functions given in Eq. (A. 21). The formula below for the decay amplitude involves a radial integral over the wave function. The Yukawa-like form containing the lepton mass m arises from the folding into the the amplitude of the lepton exchange that appears in the annihilation Feynman diagram. This amplitude gives the leading order correct result $\Gamma = m\alpha^5/2$ for the usual ground state positronium decay amplitude,

$$\mathcal{F} = \frac{e^2}{m(m+w/2)} \sqrt{2}(2\pi)^{3/2} \int_0^\infty dr r^2 j_1(wr/2) \left[\left(\frac{\exp(-mr)}{r} \right)' \left[(w/2 + m)^2 \psi(r) - \frac{\psi'(r)}{r} \right] \right]. \tag{44}$$

Using

$$\begin{aligned}
\psi(r) &= \frac{(m\sqrt{2})^{3/2}}{\sqrt{4\pi(1-\sqrt{1-4\alpha^2})!}} (\sqrt{2}rm)^{-(1-\sqrt{1-4\alpha^2})/2} \exp(-mr/\sqrt{2}), \\
\frac{\psi'(r)}{r} &= \psi(r) (-m/\sqrt{2} - (1-\sqrt{1-4\alpha^2})/2r), \\
\left(\frac{\exp(-mr)}{r} \right)' &= \frac{\exp(-mr)}{r} (-m - 1/r),
\end{aligned} \tag{45}$$

leads to

$$\mathcal{F} = -\frac{e^2}{m(m+m/\sqrt{2})}\sqrt{2}(2\pi)^{3/2}\frac{(m\sqrt{2})^{3/2}}{\sqrt{4\pi(1-\sqrt{1-4\alpha^2})!}}\int_0^\infty dr\left(\frac{2\sin(mr/\sqrt{2})}{m^2r^2}-\frac{\sqrt{2}\cos(mr/\sqrt{2})}{mr}\right) \\ (\sqrt{2}rm)^{-(1-\sqrt{(1-4\alpha^2)}/2)}\exp(-(m+m/\sqrt{2})r)(mr+1)\left[(m/\sqrt{2}+m)^2+m/r\sqrt{2}+(1-\sqrt{(1-4\alpha^2)}/2r^2)\right]. \quad (46)$$

We let $x = mr/\sqrt{2}$, and then we have

$$\mathcal{F} = -\frac{e^2}{(m+m/\sqrt{2})}2(2\pi)^{3/2}\frac{(m\sqrt{2})^{3/2}}{\sqrt{4\pi(1-\sqrt{1-4\alpha^2})!}}\int_0^\infty dx\left(\frac{\sin(x)}{x^2}-\frac{\cos(x)}{x}\right) \\ \times (2x)^{-(1-\sqrt{(1-4\alpha^2)}/2)}\exp(-(\sqrt{2}+1)x)(\sqrt{2}x+1)\left[(1/\sqrt{2}+1)^2+1/2x+(1-\sqrt{(1-4\alpha^2)}/4x^2)\right]. \quad (47)$$

To do the integral, we let $x = y/(1-y)$ and our integral becomes

$$I = \int_0^1 \frac{dy}{(1-y)^2}\left(\frac{\sin(x)}{x^2}-\frac{\cos(x)}{x}\right) \\ \times (2x)^{-(1-\sqrt{(1-4\alpha^2)}/2)}\exp(-(\sqrt{2}+1)x)(\sqrt{2}x+1)\left[(1/\sqrt{2}+1)^2+1/2x+(1-\sqrt{(1-4\alpha^2)}/4x^2)\right] \sim 0.416. \quad (48)$$

Then with

$$\frac{e^2}{(m+m/\sqrt{2})}2(2\pi)^{3/2}\frac{(m\sqrt{2})^{3/2}}{\sqrt{4\pi(1-\sqrt{1-4\alpha^2})!}} \sim 110\alpha m^{1/2}, \quad (49)$$

we obtain

$$\mathcal{F} \sim 46\alpha m^{1/2}, \quad (50)$$

and so the annihilation rate is

$$\Gamma \sim 2100m\alpha^2 \gg \frac{m\alpha^5}{2}, \quad (51)$$

much larger than the usual positronium annihilation rate. The physical reason for this is the significantly smaller size of the peculiar positronium bound state compared with the usual state. This leads to an estimated lifetime of the order of

$$\tau_{1S_p \rightarrow 2\gamma} \sim \frac{1}{2.1 \times 10^3 m\alpha^2} \sim 10^{-21} \text{sec}. \quad (52)$$

This implies that we would see 4γ in two sequential decays from the usual 1S_0 positronium state as the signature of the production and the decay of the peculiar 1S_0 state. It is likely that the two annihilation photons would have energy ranges that would be distinct from the energy range of the meta-stable decay photons. The occurrence of the peculiar state will be signaled by a total of four photons, with two photons bunching at 150 keV and two at 360 keV. Thus, by considerations of Eqs. (43) and (52), a measurement of the decay rate of the usual 1S_0 positronium state into four photons will allow the determination of P_{up} and the position of the peculiar 1S_0 state, if the electron and positron are point particles. Failure to find the peculiar state at the predicted energy would imply that either electron and positron are not point-like or P_{up} is too small. The absence of a point-like nature can arise from the electron having a structure or to other unknown interactions leading to overall less attractive interaction potentials that do not give quantum mechanically acceptable double roots of the leading short distance behavior. It is unlikely that these would result from QED higher-order corrections beyond $-\alpha^2/r^2$ due to the small value of α .

V. DISCUSSIONS AND CONCLUSIONS

Is the electron a point particle? To answer such a question, we have examined the consequences of such a property in bound states of an electron and a positron. We note that in the relativistic treatment of the electron and the positron as point particles in QED using the Two-Body Dirac equations in constraint dynamics, the magnetic interaction between e^+ and e^- in the $1S_0$ state is very large and attractive at short distance and cancels the large short distance repulsion arising from the Darwin interaction. As a consequence, the interaction at short distances behaves as $-\alpha^2/r^2$ and admits two physically allowed solutions. There is a peculiar $1S_0$ bound-state solution that has a very large binding energy of about 300 keV, in addition to the usual positronium solution with a binding energy of 6.8 eV.

We propose a search for the existence of this peculiar $1S_0$ state by looking for a four-photon decay of the usual positronium $1S_0$ state. Specifically, we envisage that the peculiar sector may be admixed with the usual sector with a mixing probability P_{up} , yet to be determined. Subsequent decay of the state to the lower peculiar state at $\sqrt{2}m$ and the prompt annihilation of the peculiar state will result in four photons, with two energies bunching at 150 keV and two at 360 keV. We estimate that the usual ground singlet state $1S_u$ state can undergo a meta-stable two photon decay with a branching ratio of about $0.152P_{up}$ compared to the dominant annihilation channel.

If the peculiar positronium ground $1S_p$ state is found, it would support the idea that the electron and positron are indeed point particles. If the peculiar ground singlet state is not found this would indicate that the electron and positron are not point particles or would alternatively set limits on the magnitude of the mixing probability P_{up} .

It is anti-intuitive that the peculiar ground state has a binding energy that does not correspond to the usual non-relativistic limit of order α^2 binding energy. Instead, its binding energy of about 300 keV is huge on atomic scales. There is some historical precedent for such anti-intuitive behavior and that was the existence of negative energy solutions of the Dirac equation for the hydrogen atom, which clearly are not physically meaningful in the non-relativistic approximation. Of course, since then with QFT being based only on positive energy particles [4], the hole model and negative energy states were discarded. Nevertheless, the signature of the 4 photon decay of the usual positronium to the peculiar ground state and its subsequent annihilation would be striking and give strong direct evidence of the point-like nature of the electron and positron.

On the other hand, the failure to find the peculiar state may provide an experimental limit on the point nature of the particles and may stimulate the search for a description of the structure of the electron. A proper description of an electron structure may help resolve the problem of infinite subtractions and infinite renormalization in QED because these large quantities are limited by the length scale of the electron structure.

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Appendix A: Details on the $1S_u \rightarrow 1S_p + 2\gamma$ Decay of the Usual Positronium Ground State

We examine the transition of the usual $1S_0$ ground state of positronium (designated by $1S_u$) into the peculiar $1S_0$ ground state (designated by $1S_p$) by emitting two photons. The Golden Rule in this case takes the form [34]

$$d^3w = 2\pi |T_{fi}|^2 d^3k_1 d^3k_2 d^3p_{1S_p} \delta(E_{1S_u} - E_{1S_p} - \hbar\omega_1 - \hbar\omega_2) \delta(\mathbf{0} - \mathbf{p}_{1S_p} - \mathbf{k}_1 - \mathbf{k}_2) \quad (\text{A. 1})$$

for the emission of photons characterized by (\mathbf{k}_1, α_1) and (\mathbf{k}_2, α_2) in the $1S_u$ rest frame. This is a very violent decay, and we cannot simply assume that the $1S_p$ is created at rest. There could be significant recoil and non-collinear alignment of the two photons. In the c.m. system of the initial positronium atom $E_{1S_p} = \sqrt{(\sqrt{2}m)^2 + \mathbf{p}_{1S_p}^2}$ and performing the $d^3p_{1S_p}$ integral gives

$$d^2w = 2\pi |T_{fi}(\omega_1, \omega_2)|^2 d^3k_1 d^3k_2 \delta(2m - E_{1S_p} - \omega_1 - \omega_2) + O(\alpha^2), \quad (\text{A. 2})$$

in which

$$\begin{aligned}
E_{1S_p} &= \sqrt{2m^2 + (\mathbf{k}_1 + \mathbf{k}_2)^2} + O(\alpha^2) \\
&= \sqrt{2m^2 + \omega_1^2 + \omega_2^2 + 2\omega_1\omega_2 \cos \theta_{12}} + O(\alpha^2), \\
\cos \theta &= \cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2 \cos(\phi_1 - \phi_2).
\end{aligned} \tag{A. 3}$$

With

$$\int \int d^3 k_1 d^3 k_2 = \int_0^\infty d\omega_1 \int_0^\infty d\omega_2 \omega_1^2 \omega_2^2 \int_0^\pi \int_0^\pi \sin \theta_1 \sin \theta_2 d\theta_1 d\theta_2 \int_0^{2\pi} \int_0^{2\pi} d\phi_1 d\phi_2. \tag{A. 4}$$

we perform the ω_2 integral using

$$\begin{aligned}
\delta(2m - E_{1S_p} - \omega_1 - \omega_2) &= \delta(f(\omega_2)) = \frac{\delta(\omega_2 - \omega_{20})}{|f'(\omega_2)|}, \\
f(\omega_2) &= 2m - \sqrt{2m^2 + \omega_1^2 + \omega_2^2 + 2\omega_1\omega_2 \cos \theta_{12}} - \omega_1 - \omega_2, \\
\omega_{20} &= \frac{2m\omega_1 - m^2}{(\omega_1(1 - \cos \theta_{12}) - 2m)}, \\
f'(\omega_2) &= -\frac{2m - \omega_1(1 - \cos \theta_{12})}{(2m - \omega_1 - \omega_2)},
\end{aligned} \tag{A. 5}$$

or

$$\left| \frac{1}{f'(\omega_{20})} \right| = \left| \frac{((2m - \omega_1)\omega_1(1 - \cos \theta_{12}) - 3m^2)}{(2m - \omega_1(1 - \cos \theta_{12}))^2} \right| \equiv g(\omega_1). \tag{A. 6}$$

Now, we must have $\omega_{20} = (2m\omega_1 - m^2)/[\omega_1(1 - \cos \theta_{12}) - 2m] > 0$. Thus we must have that either

$$\begin{aligned}
\omega_1 &> \frac{m}{2} \equiv \omega_0, \\
\text{and } \omega_1 &> \frac{2m}{(1 - \cos \theta_{12})} \equiv \tilde{\omega}_0,
\end{aligned} \tag{A. 7}$$

or

$$\begin{aligned}
\omega_1 &< \frac{m}{2} \equiv \omega_0, \\
\text{and } \omega_1 &< \frac{2m}{(1 - \cos \theta_{12})} \equiv \tilde{\omega}_0.
\end{aligned} \tag{A. 8}$$

The first of these conditions is not possible because it would allow an ω_1 that is not bounded and would clearly not satisfy energy conservation. Clearly $\omega_0, \tilde{\omega}_0 > 0$. Let us compare $\omega_0, \tilde{\omega}_0$. We find $\omega_0 < \tilde{\omega}_0$ would be true if

$$\frac{4}{(1 - \cos \theta_{12})} > 1, \tag{A. 9}$$

which is true for all $\cos \theta_{12}$. Thus, $\omega_0 < \tilde{\omega}_0$ implies second set of inequalities are true if $\omega_1 < \frac{m}{2}$. Thus we have

$$\begin{aligned}
&\int \int d^3 k_1 d^3 k_2 2\pi |T_{fi}(\omega_1, \omega_2)|^2 \delta(2m - E_{1S_p} - \omega_1 - \omega_2) \\
&= \int_0^{m/2} d\omega_1 \omega_1^2 g(\omega_1) \int_0^\pi \int_0^\pi \sin \theta_1 \sin \theta_2 d\theta_1 d\theta_2 \\
&\times \int_0^{2\pi} \int_0^{2\pi} d\phi_1 d\phi_2 \left(\frac{2m\omega_1 - m^2}{(\omega_1(1 - \cos \theta_{12}) - 2m)} \right)^2 2\pi |T_{fi}(\omega_1, \frac{2m\omega_1 - m^2}{(\omega_1(1 - \cos \theta_{12}) - 2m)})|^2.
\end{aligned} \tag{A. 10}$$

Our desired matrix element is

$$T_{fi} = \langle 1S_p \gamma \gamma | H_{int} | 1S_u \rangle M_{\zeta \zeta'}. \tag{A. 11}$$

The interaction field theoretic Hamiltonian is

$$H_{int} = -\frac{1}{4\mu} [e(\mathbf{p} \cdot \mathbf{A}(\mathbf{x}_1, t) + \mathbf{A}(\mathbf{x}_1, t) \cdot \mathbf{p}) + e^2 \mathbf{A}^2(\mathbf{x}_1)] \quad (\text{A. 12})$$

$$- \frac{1}{4\mu} [e(\mathbf{p} \cdot \mathbf{A}(\mathbf{x}_2, t) + \mathbf{A}(\mathbf{x}_2, t) \cdot \mathbf{p}) + e^2 \mathbf{A}^2(\mathbf{x}_2)],$$

in which

$$\mathbf{A}(\mathbf{x}, t) = \frac{1}{(2\pi)^{3/2}} \sum_{\mathbf{k}} \sum_{\alpha} \sqrt{\frac{1}{2\omega}} [a_{\mathbf{k},\alpha}(0) \varepsilon^{(\alpha)} e^{i\mathbf{k} \cdot \mathbf{x} - i\omega t} + a_{\mathbf{k},\alpha}^\dagger(0) \varepsilon^{(\alpha)} e^{-i\mathbf{k} \cdot \mathbf{x} + i\omega t}]. \quad (\text{A. 13})$$

The only terms of H_{int} that contribute are

$$H_{int} = -\frac{e}{2\mu} \sum_{\mathbf{k},\alpha} \sqrt{\frac{1}{2\omega_{\mathbf{k}}(2\pi)^3}} \left\{ \frac{1}{2} \varepsilon^{(\alpha)} \cdot [\mathbf{p} a_{\mathbf{k},\alpha}^\dagger e^{-i\mathbf{k} \cdot \mathbf{x}_1} + a_{\mathbf{k},\alpha}^\dagger e^{-i\mathbf{k} \cdot \mathbf{x}_1} \mathbf{p}] \right.$$

$$+ \frac{1}{2} \varepsilon^{(\alpha)} \cdot [\mathbf{p} \varepsilon^{(\alpha)} a_{\mathbf{k},\alpha}^\dagger e^{-i\mathbf{k} \cdot \mathbf{x}_2} + a_{\mathbf{k},\alpha}^\dagger e^{-i\mathbf{k} \cdot \mathbf{x}_2} \mathbf{p}] \left. \right\}$$

$$+ \frac{e^2}{m} \sum_{\mathbf{k},\alpha} \sum_{\mathbf{k}',\alpha'} \frac{1}{2\sqrt{\omega_{\mathbf{k}}\omega_{\mathbf{k}'}}(2\pi)^3} \varepsilon^{(\alpha)} \cdot \varepsilon^{(\alpha')}$$

$$\times \left\{ \frac{1}{2} \left(a_{\mathbf{k},\alpha}^\dagger a_{\mathbf{k}',\alpha'}^\dagger e^{-i(\mathbf{k}+\mathbf{k}') \cdot \mathbf{x}_1} \right) + \frac{1}{2} \left(a_{\mathbf{k},\alpha}^\dagger a_{\mathbf{k}',\alpha'}^\dagger e^{-i(\mathbf{k}+\mathbf{k}') \cdot \mathbf{x}_2} \right) \right\}$$

$$= H_{I1} + H_{I2}. \quad (\text{A. 14})$$

H_{I1} creates only one photon in first order, so we must use second order perturbation for this term. H_{I2} creates two photons, so we need only consider its first order contribution.

$$T_{fi} = M_{\zeta\zeta'} \langle 1S_p | H_{I2} | 1S_u \rangle + M_{\zeta\zeta'} \sum_I \frac{\langle 1S_p | H_{I1} | I \rangle \langle I | H_{I1} | 1S_u \rangle}{E_{1S_u} - E_I}$$

$$= \left(\frac{e^2}{m} \right) \left(\frac{M_{\zeta\zeta'}}{2(2\pi)^3 \sqrt{\omega_1 \omega_2}} \right) \left[\varepsilon^{(\alpha_1)} \cdot \varepsilon^{(\alpha_2)} \langle 1S_p | \frac{1}{2} e^{-i(\mathbf{k}_1+\mathbf{k}_2) \cdot \mathbf{x}_1} + \frac{1}{2} e^{-i(\mathbf{k}_1+\mathbf{k}_2) \cdot \mathbf{x}_2} | 1S_u \rangle \right.$$

$$+ \frac{M_{\zeta\zeta'}}{m} \sum_I \frac{1}{E_{1S_u} - E_I - \hbar\omega_1} \left(\langle 1S_p | \varepsilon^{(\alpha_2)} \cdot \left(\frac{1}{2} [\mathbf{p} e^{-i\mathbf{k}_2 \cdot \mathbf{x}_1} + e^{-i\mathbf{k}_2 \cdot \mathbf{x}_1} \mathbf{p}] + \frac{1}{2} [\mathbf{p} e^{-i\mathbf{k}_2 \cdot \mathbf{x}_2} + e^{-i\mathbf{k}_2 \cdot \mathbf{x}_2} \mathbf{p}] \right) | I \rangle \right.$$

$$\times \langle I | \varepsilon^{(\alpha_1)} \cdot \left(\frac{1}{2} [\mathbf{p} e^{-i\mathbf{k}_1 \cdot \mathbf{x}_1} + e^{-i\mathbf{k}_1 \cdot \mathbf{x}_1} \mathbf{p}] + \frac{1}{2} [\mathbf{p} e^{-i\mathbf{k}_1 \cdot \mathbf{x}_2} + e^{-i\mathbf{k}_1 \cdot \mathbf{x}_2} \mathbf{p}] \right) | 1S_u \rangle \left. \right)$$

$$+ \frac{M_{\zeta\zeta'}}{m} \sum_I \frac{1}{E_{1S_u} - E_I - \hbar\omega_2} \left(\langle 1S_p | \varepsilon^{(\alpha_1)} \cdot \left(\frac{1}{2} [\mathbf{p} e^{-i\mathbf{k}_1 \cdot \mathbf{x}_1} + e^{-i\mathbf{k}_1 \cdot \mathbf{x}_1} \mathbf{p}] + \frac{1}{2} [\mathbf{p} e^{-i\mathbf{k}_1 \cdot \mathbf{x}_2} + e^{-i\mathbf{k}_1 \cdot \mathbf{x}_2} \mathbf{p}] \right) | I \rangle \right.$$

$$\times \langle I | \varepsilon^{(\alpha_2)} \cdot \left(\frac{1}{2} [\mathbf{p} e^{-i\mathbf{k}_2 \cdot \mathbf{x}_1} + e^{-i\mathbf{k}_2 \cdot \mathbf{x}_1} \mathbf{p}] + \frac{1}{2} [\mathbf{p} e^{-i\mathbf{k}_2 \cdot \mathbf{x}_2} + e^{-i\mathbf{k}_2 \cdot \mathbf{x}_2} \mathbf{p}] \right) | 1S_u \rangle \left. \right) \Bigg], \quad (\text{A. 15})$$

in which

$$E = b^2/2\mu = b^2/m. \quad (\text{A. 16})$$

In creating two photons, the second order contributions includes the emission of (\mathbf{k}_1, α_1) followed by the emission of (\mathbf{k}_2, α_2) and also the emission of (\mathbf{k}_2, α_2) followed by the emission of (\mathbf{k}_1, α_1) . We use the dipole approximation⁶,

$$e^{-i\mathbf{k}_i \cdot \mathbf{x}_i} \simeq 1. \quad (\text{A. 17})$$

⁶ We compute $k \sim m(1 - \sqrt{2}/2)$, and with $\psi_{1S_p} \sim \exp(-mr)$, and so $kr \sim (1 - \sqrt{2}/2) \sim 0.3$ and justification for the dipole approximation is not as clear cut as with the meta-stable decay of the usual state where $k \sim m\alpha^2$, $\psi_{1S_u} \sim \exp(-m\alpha r)$, $kr \sim \alpha$ but for our purposes it is sufficient.

In that case in terms of the matrix elements of the relative coordinate \mathbf{x} , we have

$$\begin{aligned}\langle 1S_p | \mathbf{p} \cdot \varepsilon^{(\alpha_2)} | I \rangle &= -\mu i \langle 1S_p | [\mathbf{r}, H_0] \cdot \varepsilon^{(\alpha_2)} | I \rangle = \frac{1}{2} i m (E_{1S_p} - E_I) \langle 1S_p | \mathbf{x} | I \rangle \cdot \varepsilon^{(\alpha_2)}, \\ \langle I | \mathbf{p} \cdot \varepsilon^{(\alpha_1)} | 1S_u \rangle &= \frac{1}{2} i m (E_I - E_{1S_u}) \langle I | \mathbf{x} | 1S_u \rangle \cdot \varepsilon^{(\alpha_1)},\end{aligned}\quad (\text{A. 18})$$

so that

$$\begin{aligned}T_{fi} &\simeq \left(\frac{e^2}{m} \frac{1}{2(2\pi)^3 \sqrt{\omega_1 \omega_2}} \right) M_{\zeta \zeta'} \left[\varepsilon^{(\alpha_1)} \cdot \varepsilon^{(\alpha_2)} \langle 1S_p | 1S_u \rangle \right. \\ &\quad - m \sum_I (E_{1S_p} - E_I) (E_I - E_{1S_u}) \left(\frac{\langle 1S_p | \mathbf{x} | I \rangle \cdot \varepsilon^{(\alpha_2)} \langle I | \mathbf{x} | 1S_u \rangle \cdot \varepsilon^{(\alpha_1)}}{E_{1S_u} - E_I - \hbar \omega_1} \right. \\ &\quad \left. \left. + \frac{\langle 1S_p | \mathbf{x} | I \rangle \cdot \varepsilon^{(\alpha_1)} \langle I | \mathbf{x} | 1S_u \rangle}{E_{1S_u} - E_I - \hbar \omega_2} \right) \right].\end{aligned}\quad (\text{A. 19})$$

The allowed dipole transitions are ($l_f = l_i \pm 1$; $m_f = m_i, m_i \pm 1$), so from parity considerations only $l = 1, 3, \dots$ intermediate states yield non-vanishing contributions. The decay from $1S_u$ to $1S_p$ consists of a direct term and a combined transition, first from the $1S_u$ state to a virtual P state, then from the P state to the peculiar ground state.

To get an estimate of the relative size of the direct and virtual contributions we consider only the lowest lying P state and neglect the polarization factors and approximate $\mathbf{r} \rightarrow z\hat{\mathbf{k}}$. We thus approximate the amplitude by

$$\begin{aligned}T_{fi} &\simeq \left(\frac{e^2}{m} \frac{1}{2(2\pi)^3 \sqrt{\omega_1 \omega_2}} \right) M_{\zeta \zeta'} \left[\langle 1S_p | 1S_u \rangle - m (E_{1S_p} - E_{2P}) (E_{2P} - E_{1S_u}) \right. \\ &\quad \times \left(\frac{\langle 1S_p | z | 2P \rangle \langle 2P | z | 1S_u \rangle}{E_{1S_u} - E_{2P} - \hbar \omega_1} + \frac{\langle 1S_p | z | 2P \rangle \langle 2P | z | 1S_u \rangle}{E_{1S_u} - E_{2P} - \hbar \omega_2} \right) \Big].\end{aligned}\quad (\text{A. 20})$$

Given [26]

$$\begin{aligned}\psi_{1S_u} &= \frac{1}{\sqrt{4\pi}} \left[\left(\frac{2\varepsilon_{w+}\alpha}{n'_+} \right)^3 \frac{n_r!}{2n'_+(n'_+ + \lambda_+)!} \right]^{1/2} \exp\left(-\frac{\varepsilon_{w+}\alpha r}{n'_+}\right) \left(\frac{2\varepsilon_{w+}\alpha r}{n'_+} \right)^{\lambda_{\pm}}, \\ n'_+ &= (1 + \sqrt{1 - 4\alpha^2})/2 \sim 1, \quad \varepsilon_{w+} \sim m/2; \\ (n'_+ + \lambda_+)! &= (2\lambda_+ + 2 - 1)! = (\sqrt{1 - 4\alpha^2})! \sim 1, \\ \psi_{1S_p} &= \frac{1}{\sqrt{4\pi}} \left[\left(\frac{2\varepsilon_{w-}\alpha}{n'_-} \right)^3 \frac{n_r!}{2n'_-(n'_- + \lambda_-)!} \right]^{1/2} \exp\left(-\frac{\varepsilon_{w-}\alpha r}{n'_-}\right) \left(\frac{2\varepsilon_{w-}\alpha r}{n'_-} \right)^{\lambda_{\pm}}, \\ n'_- &= (1 - \sqrt{1 - 4\alpha^2})/2 \sim \alpha^2, \quad \varepsilon_{w-} \sim m\alpha/\sqrt{2}; \\ (n'_- + \lambda_-)! &= (2\lambda_- + 2 - 1)! = (-\sqrt{1 - 4\alpha^2})! \sim (-1 + 2\alpha^2)! = \frac{(2\alpha^2)!}{2\alpha^2} \sim \frac{1}{2\alpha^2},\end{aligned}\quad (\text{A. 21})$$

so that

$$\begin{aligned}\psi_{1S_u} &\sim \frac{1}{\sqrt{\pi}} \frac{(m\alpha)^{3/2}}{2\sqrt{2}} \exp\left(-\frac{m\alpha r}{2}\right), \\ \psi_{1S_p} &\sim \frac{1}{\sqrt{4\pi}} \left[\left(\sqrt{2}m \right)^3 \right]^{1/2} \exp\left(-\frac{mr}{\sqrt{2}}\right) \left(\sqrt{2}mr \right)^{(-1+\alpha^2)},\end{aligned}\quad (\text{A. 22})$$

and hence with $(\frac{1}{\sqrt{2}} + \alpha/2)mr = x$, we have

$$\begin{aligned}\langle 1S_p | 1S_u \rangle &\sim \frac{(m\alpha)^{3/2}}{4\pi\sqrt{2}} \left(\sqrt{2}m \right)^{3/2} \int d^3r \exp\left(-\left(\frac{1}{\sqrt{2}} + \alpha/2\right)mr\right) \left(\sqrt{2}mr \right)^{(-1+\alpha^2)} \\ &\sim \frac{2^{1/4}\alpha^{3/2}}{4\pi} 2\sqrt{2} \times 4\pi \times \frac{1}{\sqrt{2}} \int x dx \exp(-x) = 2^{5/4}\alpha^{3/2}.\end{aligned}\quad (\text{A. 23})$$

Next consider the matrix elements $\langle 1S_p | z | 2P(m=0) \rangle$ and $\langle 2P(m=0) | z | 1S_u \rangle$. The second one involves only usual states and roughly just one size (the Bohr radius). The $2P(m=0)$ state is

$$\psi_{2P0}(\mathbf{r}) = \frac{1}{4\sqrt{2\pi}} \left(\frac{m\alpha}{2} \right)^{3/2} \frac{rm\alpha}{2} \exp(-rm\alpha/4) \cos \theta.$$

Thus

$$\begin{aligned} \langle 2P(m=0) | r \cos \theta | 1S_u \rangle &= \frac{1}{4\sqrt{2\pi}} \left(\frac{m\alpha}{2} \right)^{3/2} \frac{1}{\sqrt{\pi}} \frac{(m\alpha)^{3/2}}{2\sqrt{2}} \\ &\times \int d^3r \exp\left(-\frac{m\alpha r}{2}\right) \frac{rm\alpha}{2} \exp(-rm\alpha/4) r \cos^2 \theta \\ &= \frac{1}{\sqrt{2}} \frac{64}{729m\alpha} \int_0^\infty dx x^4 \exp(-x) \sim \frac{1.5}{m\alpha}. \end{aligned} \quad (\text{A. 24})$$

Using $(1/\sqrt{2} + \alpha/4)mr = x$, the second one is

$$\begin{aligned} \langle 1S_p | z | 2P \rangle &= \frac{1}{8\sqrt{2\pi}} \left(\frac{m\alpha}{2} \right)^{3/2} \frac{1}{\sqrt{\pi}} \left(\sqrt{2}m \right)^{3/2} \int d^3r r \exp\left(-\frac{mr}{\sqrt{2}}\right) \left(\sqrt{2}mr \right)^{(-1+\alpha^2)} \\ &\times \frac{rm\alpha}{2} \exp(-rm\alpha/4) \cos^2 \theta \\ &= \frac{(m^3\alpha^{3/2})\alpha}{2^{3/4}24 \left[\left(\frac{1}{\sqrt{2}} + \alpha/4 \right) m \right]^4} \int dx x^3 \exp(-x) \sim \frac{\alpha^{5/2}}{2^{3/4}m}. \end{aligned} \quad (\text{A. 25})$$

Combining all factors we have in terms of orders of α

$$\begin{aligned} T_{fi} &\simeq \left(\frac{e^2}{m} \frac{1}{2(2\pi)^3 \sqrt{\omega_1 \omega_2}} \right) M_{\zeta\zeta'} \left[2^{5/4} \alpha^{3/2} - m(E_{1S_p} - E_{2P})(E_{2P} - E_{1S_u}) \right. \\ &\quad \times \left. \frac{1.5}{m\alpha} \frac{\alpha^{5/2}}{2^{3/4}m} \left(\frac{1}{E_{1S_u} - E_{2P} - \hbar\omega_1} + \frac{1}{E_{1S_u} - E_{2P} - \hbar\omega_2} \right) \right]. \end{aligned} \quad (\text{A. 26})$$

From

$$\begin{aligned} E_{2P} - E_{1S_u} &\sim m\alpha^2, \\ E_{1S_p} - E_{2P} &\sim m \left(2 - \sqrt{2} \right), \end{aligned} \quad (\text{A. 27})$$

it is clear that the second terms are order α^2 smaller than the first. We ignore these higher order pieces. Thus we approximate

$$T_{fi} \simeq \left(\frac{e^2}{m} \frac{1}{2(2\pi)^3 \sqrt{\omega_1 \omega_2}} \right) M_{\zeta\zeta'} \left[2^{5/4} \alpha^{3/2} \right]. \quad (\text{A. 28})$$

With this we compute

$$\begin{aligned} \Gamma &= \int \int d^3k_1 d^3k_2 2\pi |T_{fi}(\omega_1, \omega_2)|^2 \delta(2m - E_{1S_p} - \omega_1 - \omega_2) P_{up} \\ &= \int_0^{m/2} d\omega_1 \omega_1 g(\omega_1) \int_{-1}^{+1} dz \left(\frac{-m^2 + 2m\omega_1}{(\omega_1(1-z) - 2m)} \right) \frac{16\pi^2 \alpha^2}{m^2} \frac{\alpha^3 \sqrt{2}}{4\pi^3} P_{up}. \end{aligned} \quad (\text{A. 29})$$

We change the radial integration variable to $x = \omega_1/m$ so that, with

$$\begin{aligned} g(\omega_1) &= \left| \frac{((2m - \omega_1) \omega_1 (1 - \cos \theta_{12}) - 3m^2)}{(2m - \omega_1 (1 - \cos \theta_{12}))^2} \right| \\ &= \left| \frac{((2 - x) x (1 - z) - 3)}{(2 - x(1 - z))^2} \right|, \end{aligned} \quad (\text{A. 30})$$

we have

$$\Gamma = \frac{4m\alpha^5\sqrt{2}}{\pi} \int_0^{1/2} dx x \int_{-1}^{+1} dz \left(\frac{2x-1}{(x(1-z)-2)} \right) \left| \frac{((2-x)x(1-z)-3)}{(2-x(1-z))^2} \right| P_{up}. \quad (\text{A. 31})$$

This requires the integral

$$\int_0^{1/2} dx x \int_{-1}^{+1} dz \left(\frac{2x-1}{(x(1-z)-2)} \right) \left| \frac{((2-x)x(1-z)-3)}{(2-x(1-z))^2} \right| \sim 4.17 \times 10^{-2}. \quad (\text{A. 32})$$

The decay rate is thus

$$\begin{aligned} \Gamma &= \frac{4m\alpha^5\sqrt{2}}{\pi} 4.17 \times 10^{-2} P_{up} \\ &= \Gamma_{1S_u \rightarrow 2\gamma} 0.152 P_{up} \end{aligned} \quad (\text{A. 33})$$

and so the branching ratio is $0.152 P_{up}$.

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